$$\begin{aligned}\mathbb{E}g(X) &= \mathbb{E}\left(\sum_{k=1}^{n}\alpha_k \mathbb{I}_{B_k}(X)\right) = \sum_{k=1}^{n}\alpha_k \mathbb{E}\mathbb{I}_{B_k}(X)\\ &= \sum_{k=1}^{n}\alpha_k \int_{-\infty}^{\infty}\mathbb{I}_{B_k}(x)f(x)\,dx = \int_{-\infty}^{\infty}\sum_{k=1}^{n}\alpha_k\mathbb{I}_{B_k}(x)f(x)\,dx\\ &= \int_{-\infty}^{\infty} g(x)f(x)\,dx.\end{aligned}$$

Step 3. Nonnegative Borel-measurable functions. Just as in the proof of Theorem 1.5.1 we construct a sequence of nonnegative simple functions $0 \le g_1 \le g_2 \le \cdots \le g$ such that $\lim_{n\to\infty} g_n(x) = g(x)$ for every $x \in R$. We have already shown that

$$\mathbb{E}g_n(X) = \int_{-\infty}^{\infty} g_n(x)f(x)\,dx$$

for every n. We let $n \to \infty$, using the Monotone Convergence Theorem, Theorem 1.4.5, on both sides of the equation, to obtain (1.5.7).

Step 4. General Borel-measurable functions. Let g be a general Borel-measurable function, which can take positive and negative values. We have just proved that

$$\mathbb{E}g^+(X) = \int_{-\infty}^{\infty} g^+(x)f(x)\,dx, \quad \mathbb{E}g^-(X) = \int_{-\infty}^{\infty} g^-(x)f(x)\,dx.$$

Adding these equations, we obtain (1.5.6). If the expression in (1.5.6) is finite, we can also subtract these equations to obtain (1.5.7). □

1.6 Change of Measure

We pick up the thread of Section 3.1 of Volume I, in which we used a positive random variable Z to change probability measures on a space Ω. We need to do this when we change from the actual probability measure $\mathbb{P}$ to the risk-neutral probability measure $\widetilde{\mathbb{P}}$ in models of financial markets. When Ω is uncountably infinite and $\mathbb{P}(\omega) = \widetilde{\mathbb{P}}(\omega) = 0$ for every $\omega \in \Omega$, it no longer makes sense to write (3.1.1) of Chapter 3 of Volume I,

$$Z(\omega) = \frac{\widetilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}, \tag{1.6.1}$$

because division by zero is undefined. We could rewrite this equation as

$$Z(\omega)\mathbb{P}(\omega) = \widetilde{\mathbb{P}}(\omega), \tag{1.6.2}$$

and now we have a meaningful equation, with both sides equal to zero, but the equation tells us nothing about the relationship among $\mathbb{P}$, $\widetilde{\mathbb{P}}$, and Z. Because

$\mathbb{P}(\omega) = \widetilde{\mathbb{P}}(\omega) = 0$, the value of $Z(\omega)$ could be anything and (1.6.2) would still hold.

However, (1.6.2) does capture the spirit of what we would like to accomplish. To change from $\mathbb{P}$ to $\widetilde{\mathbb{P}}$, we need to reassign probabilities in Ω using Z to tell us where in Ω we should revise the probability upward (where $Z > 1$) and where we should revise the probability downward (where $Z < 1$). However, we should do this set-by-set, rather than ω-by-ω. The process is described by the following theorem.

Theorem 1.6.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely nonnegative random variable with $\mathbb{E}Z = 1$. For $A \in \mathcal{F}$, define*

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)\, d\mathbb{P}(\omega). \tag{1.6.3}$$

Then $\widetilde{\mathbb{P}}$ is a probability measure. Furthermore, if X is a nonnegative random variable, then

$$\widetilde{\mathbb{E}}X = \mathbb{E}[XZ]. \tag{1.6.4}$$

If Z is almost surely strictly positive, we also have

$$\mathbb{E}Y = \widetilde{\mathbb{E}}\left[\frac{Y}{Z}\right] \tag{1.6.5}$$

for every nonnegative random variable Y.

The $\widetilde{\mathbb{E}}$ appearing in (1.6.4) is expectation under the probability measure $\widetilde{\mathbb{P}}$ (i.e., $\widetilde{\mathbb{E}}X = \int_\Omega X(\omega)\, d\widetilde{\mathbb{P}}(\omega)$).

Remark 1.6.2. Suppose X is a random variable that can take both positive and negative values. We may apply (1.6.4) to its positive and negative parts $X^+ = \max\{X, 0\}$ and $X^- = \max\{-X, 0\}$, and then subtract the resulting equations to see that (1.6.4) holds for this X as well, provided the subtraction does not result in an $\infty - \infty$ situation. The same remark applies to (1.6.5).

PROOF OF THEOREM 1.6.1: According to Definition 1.1.2, to check that $\widetilde{\mathbb{P}}$ is a probability measure, we must verify that $\widetilde{\mathbb{P}}(\Omega) = 1$ and that $\widetilde{\mathbb{P}}$ is countably additive. We have by assumption

$$\widetilde{\mathbb{P}}(\Omega) = \int_\Omega Z(\omega)\, d\mathbb{P}(\omega) = \mathbb{E}Z = 1.$$

For countable additivity, let $A_1, A_2, \ldots$ be a sequence of disjoint sets in $\mathcal{F}$, and define $B_n = \cup_{k=1}^n A_k$, $B_\infty = \cup_{k=1}^\infty A_k$. Because

$$\mathbb{I}_{B_1} \le \mathbb{I}_{B_2} \le \mathbb{I}_{B_3} \le \cdots$$

and $\lim_{n\to\infty} \mathbb{I}_{B_n} = \mathbb{I}_{B_\infty}$, we may use the Monotone Convergence Theorem, Theorem 1.4.5, to write

$$\widetilde{\mathbb{P}}(B_\infty) = \int_\Omega \mathbb{I}_{B_\infty}(\omega) Z(\omega)\, d\mathbb{P}(\omega) = \lim_{n\to\infty} \int_\Omega \mathbb{I}_{B_n}(\omega) Z(\omega)\, d\mathbb{P}(\omega).$$

But $\mathbb{I}_{B_n}(\omega) = \sum_{k=1}^n \mathbb{I}_{A_k}(\omega)$, and so

$$\int_\Omega \mathbb{I}_{B_n}(\omega) Z(\omega)\, d\mathbb{P}(\omega) = \sum_{k=1}^n \int_\Omega \mathbb{I}_{A_k}(\omega) Z(\omega)\, d\mathbb{P}(\omega) = \sum_{k=1}^n \widetilde{\mathbb{P}}(A_k).$$

Putting these two equations together, we obtain the countable additivity property

$$\widetilde{\mathbb{P}}\left(\bigcup_{k=1}^\infty A_k\right) = \lim_{n\to\infty} \sum_{k=1}^n \widetilde{\mathbb{P}}(A_k) = \sum_{k=1}^\infty \widetilde{\mathbb{P}}(A_k).$$

Now suppose X is a nonnegative random variable. If X is an indicator function $X = \mathbb{I}_A$, then

$$\widetilde{\mathbb{E}}X = \widetilde{\mathbb{P}}(A) = \int_\Omega \mathbb{I}_A(\omega) Z(\omega)\, d\mathbb{P}(\omega) = \mathbb{E}\left[\mathbb{I}_A Z\right] = \mathbb{E}\left[XZ\right],$$

which is (1.6.4). We finish the proof of (1.6.4) using the standard machine developed in Theorem 1.5.1. When $Z > 0$ almost surely, $\frac{Y}{Z}$ is defined and we may replace X in (1.6.4) by $\frac{Y}{Z}$ to obtain (1.6.5). □

Definition 1.6.3. *Let Ω be a nonempty set and $\mathcal{F}$ a σ-algebra of subsets of Ω. Two probability measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ are said to be* equivalent *if they agree which sets in $\mathcal{F}$ have probability zero.*

Under the assumptions of Theorem 1.6.1, including the assumption that $Z > 0$ almost surely, $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent. Suppose $A \in \mathcal{F}$ is given and $\mathbb{P}(A) = 0$. Then the random variable $\mathbb{I}_A Z$ is $\mathbb{P}$ almost surely zero, which implies

$$\widetilde{\mathbb{P}}(A) = \int_\Omega \mathbb{I}_A(\omega) Z(\omega)\, d\mathbb{P}(\omega) = 0.$$

On the other hand, suppose $B \in \mathcal{F}$ satisfies $\widetilde{\mathbb{P}}(B) = 0$. Then $\frac{1}{Z}\mathbb{I}_B = 0$ almost surely under $\widetilde{\mathbb{P}}$, so

$$\widetilde{\mathbb{E}}\left[\frac{1}{Z}\mathbb{I}_B\right] = 0.$$

Equation (1.6.5) implies $\mathbb{P}(B) = \mathbb{E}\mathbb{I}_B = 0$. This shows that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ agree which sets have probability zero. Because the sets with probability one are complements of the sets with probability zero, $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ agree which sets have probability one as well. Because $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent, we do not need to specify which measure we have in mind when we say an event occurs *almost surely*.

In financial models, we will first set up a sample space Ω, which one can regard as the set of possible scenarios for the future. We imagine this set of possible scenarios has an actual probability measure $\mathbb{P}$. However, for purposes of pricing derivative securities, we will use a risk-neutral measure $\widetilde{\mathbb{P}}$. We will insist that these two measures are equivalent. They must agree on what is possible and what is impossible; they may disagree on how probable the possibilities are. This is the same situation we had in the binomial model; $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ assigned different probabilities to the stock price paths, but they agreed which stock price paths were possible. In the continuous-time model, after we have $\mathbb{P}$ and $\widetilde{\mathbb{P}}$, we shall determine prices of derivative securities that allow us to set up hedges that work with $\widetilde{\mathbb{P}}$-probability one. These hedges then also work with $\mathbb{P}$-probability one. Although we have used the risk-neutral probability to compute prices, we will have obtained hedges that work with probability one under the actual (and the risk-neutral) probability measure.

It is common to refer to computations done under the actual measure as computations in the *real world* and computations done under the risk-neutral measure as computations in the *risk-neutral world.* This unfortunate terminology raises the question whether prices computed in the "risk-neutral world" are appropriate for the "real world" in which we live and have our profits and losses. Our answer to this question is that *there is only one world* in the models. There is a single sample space Ω representing all possible future states of the financial markets, and there is a single set of asset prices, modeled by random variables (i.e., functions of these future states of the market). We sometimes work in this world assuming that probabilities are given by an empirically estimated actual probability measure and sometimes assuming that they are given by risk-neutral probabilities, but we do not change our view of the world of possibilities. A hedge that works almost surely under one assumption of probabilities works almost surely under the other assumption as well, since the probability measures agree which events have probability one.

The change of measure discussed in Section 3.1 of Volume I is the special case of Theorem 1.6.1 for finite probability spaces, and Example 3.1.2 of Chapter 3 of Volume I provides a case with explicit formulas for $\mathbb{P}$, $\widetilde{\mathbb{P}}$, and Z when the expectations are sums. We give here two examples on uncountable probability spaces.

Example 1.6.4. Recall Example 1.2.4 in which $\Omega = [0,1]$, $\mathbb{P}$ is the uniform (i.e., Lebesgue) measure, and

$$\widetilde{\mathbb{P}}[a,b] = \int_a^b 2\omega\, d\omega = b^2 - a^2,\ 0 \le a \le b \le 1. \tag{1.2.2}$$

We may use the fact that $\mathbb{P}(d\omega) = d\omega$ to rewrite (1.2.2) as

$$\widetilde{\mathbb{P}}[a,b] = \int_{[a,b]} 2\omega\, d\mathbb{P}(\omega). \tag{1.2.2$'$}$$

Because $\mathcal{B}[0,1]$ is the σ-algebra generated by the closed intervals (i.e., begin with the closed intervals and put in all other sets necessary in order to have a

σ-algebra), the validity of (1.2.2)$'$ for all closed intervals $[a,b] \subset [0,1]$ implies its validity for all Borel subsets of $[0,1]$:

$$\widetilde{\mathbb{P}}(B) = \int_B 2\omega \, d\mathbb{P}(\omega) \text{ for every Borel set } B \subset \mathbb{R}.$$

This is (1.6.3) with $Z(\omega) = 2\omega$.

Note that $Z(\omega) = 2\omega$ is strictly positive almost surely ($\mathbb{P}\{0\} = 0$), and

$$\widetilde{\mathbb{E}}Z = \int_0^1 2\omega \, d\omega = 1.$$

According to (1.6.4), for every nonnegative random variable $X(\omega)$, we have the equation

$$\int_0^1 X(\omega) \, d\widetilde{\mathbb{P}}(\omega) = \int_0^1 X(\omega) \cdot 2\omega \, d\omega.$$

This suggests the notation

$$d\widetilde{\mathbb{P}}(\omega) = 2\omega \, d\omega = 2\omega \, d\mathbb{P}(\omega). \tag{1.6.6}$$

□

In general, when $\mathbb{P}$, $\widetilde{\mathbb{P}}$, and Z are related as in Theorem 1.6.1, we may rewrite the two equations (1.6.4) and (1.6.5) as

$$\int_\Omega X(\omega) \, d\widetilde{\mathbb{P}}(\omega) = \int_\Omega X(\omega) Z(\omega) \, d\mathbb{P}(\omega),$$
$$\int_\Omega Y(\omega) \, d\mathbb{P}(\omega) = \int_\Omega \frac{Y(\omega)}{Z(\omega)} \, d\widetilde{\mathbb{P}}(\omega).$$

A good way to remember these equations is to formally write $Z(\omega) = \frac{d\widetilde{\mathbb{P}}(\omega)}{d\mathbb{P}(\omega)}$. Equation (1.6.6) is a special case of this notation that captures the idea behind the nonsensical equation (1.6.1) that Z is somehow a "ratio of probabilities." In Example 1.6.4, $Z(\omega) = 2\omega$ is in fact a ratio of densities, with the denominator being the uniform density 1 for all $\omega \in [0,1]$.

Definition 1.6.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\widetilde{\mathbb{P}}$ be another probability measure on $(\Omega, \mathcal{F})$ that is equivalent to $\mathbb{P}$, and let Z be an almost surely positive random variable that relates $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ via (1.6.3). Then Z is called the* Radon-Nikodým derivative *of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and we write*

$$Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}.$$

Example 1.6.6 (Change of measure for a normal random variable). Let X be a standard normal random variable defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Two ways of constructing X and $(\Omega, \mathcal{F}, \mathbb{P})$ were described in Example 1.2.6. For purposes of this example, we do not need to know the details about the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, except we note that the set Ω is necessarily uncountably infinite and $\mathbb{P}(\omega) = 0$ for every $\omega \in \Omega$.

When we say X is a standard normal random variable, we mean that

$$\mu_X(B) = \mathbb{P}\{X \in B\} = \int_B \varphi(x)\,dx \text{ for every Borel subset } B \text{ of } \mathbb{R}, \quad (1.6.7)$$

where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

is the standard normal density. If we take $B = (-\infty, b]$, this reduces to the more familiar condition

$$\mathbb{P}\{X \leq b\} = \int_{-\infty}^{b} \varphi(x)dx \text{ for every } b \in \mathbb{R}. \quad (1.6.8)$$

In fact, (1.6.8) is equivalent to the apparently stronger statement (1.6.7). Note that $\mathbb{E}X = 0$ and variance $\text{Var}(X) = \mathbb{E}(X - \mathbb{E}X)^2 = 1$.

Let θ be a constant and define $Y = X + \theta$, so that under $\mathbb{P}$, the random variable Y is normal with $\mathbb{E}Y = \theta$ and variance $\text{Var}(Y) = \mathbb{E}(Y - \mathbb{E}Y)^2 = 1$. Although it is not required by the formulas, we will assume θ is positive for the discussion below. We want to change to a new probability measure $\widetilde{\mathbb{P}}$ on Ω under which Y is a standard normal random variable. In other words, we want $\widetilde{\mathbb{E}}Y = 0$ and $\widetilde{\text{Var}}(Y) = \widetilde{\mathbb{E}}(Y - \widetilde{\mathbb{E}}Y)^2 = 1$. We want to do this not by subtracting θ away from Y, but rather by assigning less probability to those ω for which $Y(\omega)$ is sufficiently positive and more probability to those ω for which $Y(\omega)$ is negative. *We want to change the distribution of Y without changing the random variable Y.* In finance, the change from the actual to the risk-neutral probability measure changes the distribution of asset prices without changing the asset prices themselves, and this example is a step in understanding that procedure.

We first define the random variable

$$Z(\omega) = \exp\left\{-\theta X(\omega) - \frac{1}{2}\theta^2\right\} \text{ for all } \omega \in \Omega.$$

This random variable has two important properties that allow it to serve as a Radon-Nikodým derivative for obtaining a probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$:

(i) $Z(\omega) > 0$ for all $\omega \in \Omega$ ($Z > 0$ almost surely would be good enough), and
(ii) $\mathbb{E}Z = 1$.

Property (i) is obvious because Z is defined as an exponential. Property (ii) follows from the integration

$$\begin{aligned}\mathbb{E}Z &= \int_{-\infty}^{\infty} \exp\left\{-\theta x - \frac{1}{2}\theta^2\right\} \varphi(x)dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}(x^2 + 2\theta x + \theta^2)\right\} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}(x+\theta)^2\right\} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}y^2\right\} dy,\end{aligned}$$

where we have made the change of dummy variable $y = x + \theta$ in the last step. But $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}y^2\}dy$, being the integral of the standard normal density, is equal to one.

We use the random variable Z to create a new probability measure $\widetilde{\mathbb{P}}$ by adjusting the probabilities of the events in Ω. We do this by defining

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)\, d\mathbb{P}(\omega) \text{ for all } A \in \mathcal{F}. \tag{1.6.9}$$

The random variable Z has the property that if $X(\omega)$ is positive, then $Z(\omega) < 1$ (we are still thinking of θ as a positive constant). This shows that $\widetilde{\mathbb{P}}$ assigns less probability than $\mathbb{P}$ to sets on which X is positive, a step in the right direction of statistically recentering Y. We claim not only that $\widetilde{\mathbb{E}}Y = 0$ but also that, under $\widetilde{\mathbb{P}}$, Y is a standard normal random variable. To see this, we compute

$$\begin{aligned}\widetilde{\mathbb{P}}\{Y \le b\} &= \int_{\{\omega; Y(\omega) \le b\}} Z(\omega)\, d\mathbb{P}(\omega) \\ &= \int_\Omega \mathbb{I}_{\{Y(\omega) \le b\}} Z(\omega)\, d\mathbb{P}(\omega) \\ &= \int_\Omega \mathbb{I}_{\{X(\omega) \le b-\theta\}} \exp\left\{-\theta X(\omega) - \frac{1}{2}\theta^2\right\} d\mathbb{P}(\omega).\end{aligned}$$

At this point, we have managed to write $\widetilde{\mathbb{P}}\{Y \le b\}$ in terms of a function of the random variable X, integrated with respect to the probability measure $\mathbb{P}$ under which X is standard normal. According to Theorem 1.5.2,

$$\begin{aligned}\int_\Omega \mathbb{I}_{\{X(\omega) \le b-\theta\}} \exp\left\{-\theta X(\omega) - \frac{1}{2}\theta^2\right\} d\mathbb{P}(\omega) \\ = \int_{-\infty}^{\infty} \mathbb{I}_{\{x \le b-\theta\}} e^{-\theta x - \frac{1}{2}\theta^2} \varphi(x)dx \\ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b-\theta} e^{-\theta x - \frac{1}{2}\theta^2} e^{-\frac{x^2}{2}} dx\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b-\theta} e^{-\frac{1}{2}(x+\theta)^2} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-\frac{1}{2}y^2} dy,$$

where we have made the change of dummy variable $y = x + \theta$ in the last step. We conclude that

$$\widetilde{\mathbb{P}}\{Y \leq b\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-\frac{1}{2}y^2} dy,$$

which shows that Y is a standard normal random variable under the probability measure $\widetilde{\mathbb{P}}$. □

Following Corollary 2.4.6 of Chapter 2 of Volume I, we discussed how the existence of a risk-neutral measure guarantees that a financial model is free of arbitrage, the so-called *First Fundamental Theorem of Asset Pricing.* The same argument applies in continuous-time models and in fact underlies the Heath-Jarrow-Morton no-arbitrage condition for term-structure models. Consequently, we are interested in the existence of risk-neutral measures. As discussed earlier in this section, these must be equivalent to the actual probability measure. How can such probability measures $\widetilde{\mathbb{P}}$ arise? In Theorem 1.6.1, we began with the probability measure $\mathbb{P}$ and an almost surely positive random variable Z and constructed the equivalent probability measure $\widetilde{\mathbb{P}}$. It turns out that this is the only way to obtain a probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$. The proof of the following profound theorem is beyond the scope of this text.

Theorem 1.6.7 (Radon-Nikodým). *Let $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ be equivalent probability measures defined on $(\Omega, \mathcal{F})$. Then there exists an almost surely positive random variable Z such that $\mathbb{E}Z = 1$ and*

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)\, d\mathbb{P}(\omega) \text{ for every } A \in \mathcal{F}.$$

1.7 Summary

Probability theory begins with a *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ (Definition 1.1.2). Here Ω is the set of all possible outcomes of a random experiment, $\mathcal{F}$ is the collection of subsets of Ω whose probability is defined, and $\mathbb{P}$ is a function mapping $\mathcal{F}$ to $[0, 1]$. The two axioms of probability spaces are $\mathbb{P}(\Omega) = 1$ and *countable additivity*: the probability of a union of disjoint sets is the sum of the probabilities of the individual sets.

The collection of sets $\mathcal{F}$ in the preceding paragraph is a *σ-algebra*, which means that $\emptyset$ belongs to $\mathcal{F}$, the complement of every set in $\mathcal{F}$ is also in $\mathcal{F}$, and the union of any sequence of sets in $\mathcal{F}$ is also in $\mathcal{F}$. The Borel σ-algebra in $\mathbb{R}$, denoted $\mathcal{B}(\mathbb{R})$, is the smallest σ-algebra that contains all the closed interval

Then we say that X is *independent* of the event A.

Show that if X is independent of an event A, then

$$\int_A g(X(\omega))\,d\mathbb{P}(\omega) = \mathbb{P}(A) \cdot \mathbb{E}g(X)$$

for every nonnegative, Borel-measurable function g.

Exercise 1.10. Let $\mathbb{P}$ be the uniform (Lebesgue) measure on $\Omega = [0,1]$. Define

$$Z(\omega) = \begin{cases} 0 \text{ if } 0 \le \omega < \frac{1}{2}, \\ 2 \text{ if } \frac{1}{2} \le \omega \le 1. \end{cases}$$

For $A \in \mathcal{B}[0,1]$, define

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)\,d\mathbb{P}(\omega).$$

(i) Show that $\widetilde{\mathbb{P}}$ is a probability measure.
(ii) Show that if $\mathbb{P}(A) = 0$, then $\widetilde{\mathbb{P}}(A) = 0$. We say that $\widetilde{\mathbb{P}}$ is *absolutely continuous* with respect to $\mathbb{P}$.
(iii) Show that there is a set A for which $\widetilde{\mathbb{P}}(A) = 0$ but $\mathbb{P}(A) > 0$. In other words, $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are not equivalent.

Exercise 1.11. In Example 1.6.6, we began with a standard normal random variable X under a measure $\mathbb{P}$. According to Exercise 1.6, this random variable has the moment-generating function

$$\mathbb{E}e^{uX} = e^{\frac{1}{2}u^2} \text{ for all } u \in \mathbb{R}.$$

The moment-generating function of a random variable determines its distribution. In particular, any random variable that has moment-generating function $e^{\frac{1}{2}u^2}$ must be standard normal.

In Example 1.6.6, we also defined $Y = X + \theta$, where θ is a constant, we set $Z = e^{-\theta X - \frac{1}{2}\theta^2}$, and we defined $\widetilde{\mathbb{P}}$ by the formula (1.6.9):

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)\,d\mathbb{P}(\omega) \text{ for all } A \in \mathcal{F}.$$

We showed by considering its cumulative distribution function that Y is a standard normal random variable under $\widetilde{\mathbb{P}}$. Give another proof that Y is standard normal under $\widetilde{\mathbb{P}}$ by verifying the moment-generating function formula

$$\widetilde{\mathbb{E}}e^{uY} = e^{\frac{1}{2}u^2} \text{ for all } u \in \mathbb{R}.$$

Exercise 1.12. In Example 1.6.6, we began with a standard normal random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and defined the random variable $Y = X + \theta$, where θ is a constant. We also defined $Z = e^{-\theta X - \frac{1}{2}\theta^2}$ and used Z as the Radon-Nikodým derivative to construct the probability measure $\widetilde{\mathbb{P}}$ by the formula (1.6.9):

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)\,d\mathbb{P}(\omega) \text{ for all } A \in \mathcal{F}.$$

Under $\widetilde{\mathbb{P}}$, the random variable Y was shown to be standard normal.

We now have a standard normal random variable Y on the probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$, and X is related to Y by $X = Y - \theta$. By what we have just stated, with X replaced by Y and θ replaced by $-\theta$, we could define $\widehat{Z} = e^{\theta Y - \frac{1}{2}\theta^2}$ and then use $\widehat{Z}$ as a Radon-Nikodým derivative to construct a probability measure $\widehat{\mathbb{P}}$ by the formula

$$\widehat{\mathbb{P}}(A) = \int_A \widehat{Z}(\omega)\,d\widetilde{\mathbb{P}}(\omega) \text{ for all } A \in \mathcal{F},$$

so that, under $\widehat{\mathbb{P}}$, the random variable X is standard normal. Show that $\widehat{Z} = \frac{1}{Z}$ and $\widehat{\mathbb{P}} = \mathbb{P}$.

Exercise 1.13 (Change of measure for a normal random variable). A nonrigorous but informative derivation of the formula for the Radon-Nikodým derivative $Z(\omega)$ in Example 1.6.6 is provided by this exercise. As in that example, let X be a standard normal random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $Y = X + \theta$. Our goal is to define a strictly positive random variable $Z(\omega)$ so that when we set

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega)\,d\mathbb{P}(\omega) \text{ for all } A \in \mathcal{F}, \tag{1.9.4}$$

the random variable Y under $\widetilde{\mathbb{P}}$ is standard normal. If we fix $\overline{\omega} \in \Omega$ and choose a set A that contains $\overline{\omega}$ and is "small," then (1.9.4) gives

$$\widetilde{\mathbb{P}}(A) \approx Z(\overline{\omega})\mathbb{P}(A),$$

where the symbol $\approx$ means "is approximately equal to." Dividing by $\mathbb{P}(A)$, we see that

$$\frac{\widetilde{\mathbb{P}}(A)}{\mathbb{P}(A)} \approx Z(\overline{\omega})$$

for "small" sets A containing $\overline{\omega}$. We use this observation to identify $Z(\overline{\omega})$.

With $\overline{\omega}$ fixed, let $x = X(\overline{\omega})$. For $\epsilon > 0$, we define $B(x,\epsilon) = \left[x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}\right]$ to be the closed interval centered at x and having length ϵ. Let $y = x + \theta$ and $B(y,\epsilon) = \left[y - \frac{\epsilon}{2}, y + \frac{\epsilon}{2}\right]$.

(i) Show that

$$\frac{1}{\epsilon}\mathbb{P}\{X \in B(x,\epsilon)\} \approx \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{X^2(\overline{\omega})}{2}\right\}.$$

(ii) In order for Y to be a standard normal random variable under $\widetilde{\mathbb{P}}$, show that we must have

$$\frac{1}{\epsilon}\widetilde{\mathbb{P}}\{Y \in B(y,\epsilon)\} \approx \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{Y^2(\overline{\omega})}{2}\right\}.$$

(iii) Show that $\{X \in B(x,\epsilon)\}$ and $\{Y \in B(y,\epsilon)\}$ are the same set, which we call $A(\overline{\omega},\epsilon)$. This set contains $\overline{\omega}$ and is "small" when $\epsilon > 0$ is small.

(iv) Show that

$$\frac{\widetilde{\mathbb{P}}(A)}{\mathbb{P}(A)} \approx \exp\left\{-\theta X(\overline{\omega}) - \frac{1}{2}\theta^2\right\}.$$

The right-hand side is the value we obtained for $Z(\overline{\omega})$ in Example 1.6.6.

Exercise 1.14 (Change of measure for an exponential random variable). Let X be a nonnegative random variable defined on a probability space $(\Omega,\mathcal{F},P)$ with the *exponential distribution*, which is

$$\mathbb{P}\{X \le a\} = 1 - e^{-\lambda a},\ a \ge 0,$$

where λ is a positive constant. Let $\tilde{\lambda}$ be another positive constant, and define

$$Z = \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)X}.$$

Define $\widetilde{\mathbb{P}}$ by

$$\widetilde{\mathbb{P}}(A) = \int_A Z\,d\mathbb{P} \quad \text{for all } A \in \mathcal{F}.$$

(i) Show that $\widetilde{\mathbb{P}}(\Omega) = 1$.

(ii) Compute the cumulative distribution function

$$\widetilde{\mathbb{P}}\{X \le a\} \text{ for } a \ge 0$$

for the random variable X under the probability measure $\widetilde{\mathbb{P}}$.

Exercise 1.15 (Provided by Alexander Ng). Let X be a random variable on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, and assume X has a density function $f(x)$ that is positive for every $x \in \mathbb{R}$. Let g be a strictly increasing, differentiable function satisfying

$$\lim_{y\to-\infty} g(y) = -\infty, \quad \lim_{y\to\infty} g(y) = \infty,$$

and define the random variable $Y = g(X)$.

Let $h(y)$ be an arbitrary nonnegative function satisfying $\int_{-\infty}^{\infty} h(y)\,dy = 1$. We want to change the probability measure so that $h(y)$ is the density function for the random variable Y. To do this, we define

$$Z = \frac{h(g(X))g'(X)}{f(X)}.$$

(i) Show that Z is nonnegative and $\mathbb{E}Z = 1$.

Now define $\widetilde{\mathbb{P}}$ by

$$\widetilde{\mathbb{P}}(A) = \int_A Z\,d\mathbb{P} \quad \text{for all } A \in \mathcal{F}.$$

(ii) Show that Y has density h under $\widetilde{\mathbb{P}}$.